MATHEMATICS

“BRIEF CALCULUS”

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Introduction

With the goal of improving student learning, a group of seventeen mathematics faculty from eight campuses of the University of South Carolina System worked over a period of three summers (1992-1994) on a revision of the brief calculus course. (Most colleges and universities in the US have a one-semester calculus course, called "Brief Calculus" by mathematics faculty, intended for students majoring in business, humanities, and the social sciences.) For the most part, these seventeen faculty had not worked together prior to the start of the project. The project, which also addressed general education courses in three other disciplines, was supported by a grant from the Fund for the Improvement of Post-Secondary Education (FIPSE). We (the mathematics group) articulated a statement of desired student outcomes (see attached "Goals for the Brief Calculus Course"), taking account of the non-technical audience for this course. We also developed criteria to assess whether or not students achieve these goals (see attached "Criteria for Assessment of Student Learning in the Brief Calculus Course"). It is our intention that both the Goals and the Criteria should be shared with the students, and that the students should be asked to focus on these goals and criteria throughout the course.

We recognized that the traditional-lecture style course, emphasizing symbol manipulation and ignoring technology, does not promote these goals. Therefore, we wrote a set of activities in the spirit of the national calculus reform movement (see the attached Resource List of Calculus Reform Materials) involving active learning, relevant applications, group work, and the use of current technology. Each activity includes at its beginning a list of the course goals addressed, and the students are asked to respond to a self-assessment questionnaire at the conclusion of each activity, so that the students are led to reflect on what they did and to think consciously about whether or not they met these goals as measured by the criteria for these goals.

For some students this course is an absolute requirement for their desired major. Many of these students bring to the course a history of negative experiences in mathematics and a great deal of anxiety over taking a mathematics class. For the most part, these students have never felt that there was a connection between their mathematics classes and any other aspect of their lives. Our desire was to enhance student success, to make this course a gateway to and not a gatekeeper from professional opportunity, by relating the course materials to applications the students will find interesting and by presenting the material in a way that involves active participation by the students.

Our new vision for this course involves some changes in emphasis and approach. For example, we believe that students should write and speak more about mathematics than has been traditional. We also believe that the course should emphasize visualization, aided by graphing technology. Students should demonstrate understanding of the concept of a function and its graph by applying verbal descriptions as well as rule definitions, and should demonstrate understanding of first and second derivatives both geometrically and as rates of change. Thus students should be able to sketch graphs and to describe them in terms of derivatives from verbal descriptions, and should be able to write a verbal description (including statements involving first and second derivatives) if they are given a graph. In this course, the study of limits should be pursued not for its own sake but only informally as needed in context: defining the derivative, describing "end behavior" of graphs, and understanding the number e. Students should demonstrate understanding of the definite integral and its use not only through applying the Fundamental Theorem of Calculus but also through visualization as a signed area. Throughout the course, students should be actively involved in constructing their own understanding. The activities we wrote do not cover the entire course, but we believe they will make it possible for the instructor to implement this vision.

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Goals for the Brief Calculus Course

A. The student will develop mathematical modeling ability to solve real-world problems.

B. The student will demonstrate the ability to communicate effectively to various audiences.

C. The student will demonstrate the ability to use concepts and techniques of calculus.

D. The student will develop the ability to use calculus concepts to interpret quantitative information qualitatively.

E. The student will use graphs to help conceptualize and solve problems.

F. The student will develop the ability to interact effectively with a group to set goals and complete specific tasks.

Criteria for Assessment of Student Learning in the Brief Calculus Course

A. The student will develop mathematical modeling ability to solve real-world problems by:
   1. Analyzing the problem situation
   2. Making critical observations and accurate inferences and relating them to mathematical concepts
   3. Designing a solution strategy using the concepts of calculus
   4. Implementing the solution strategy using calculus techniques and appropriate technology

B. The student will show effective communication in mathematics by:
   1. Producing accurate, well-written problem solutions which
      a. use standard English
      b. employ accurate mathematical terminology and notation
      c. feature clear exposition and logical development
      d. are appropriate for the given audience
   2. Making coherent oral presentations which
      a. use standard English
      b. employ appropriate mathematical terminology
      c. show logical development
      d. employ clear visual aids
      e. feature accurate solutions
      f. display poise and confidence

C. The student will demonstrate the ability to use concepts and techniques of calculus by:
   1. Analyzing functions
   2. Computing, interpreting, and applying derivatives and integrals of functions

D. The student will develop the ability to use calculus concepts to interpret quantitative information qualitatively by:
   1. Sketching a graph with an appropriate shape from a verbal description
   2. Interpreting and evaluating the mathematical implications of various media reports

E. The student will demonstrate the ability to apply visual methods in calculus by:
   1. Using modern technology to construct graphs of functions, with an appropriate viewing window
   2. Using graphs to
      a. visualize problem situations
      b. identify possible solutions
      c. discern properties of functions

F. The student will demonstrate effective group skills by:
   1. Showing mutual respect
   2. Staying on task
   3. Identifying goals
   4. Contributing individually
   5. Asking appropriate questions within the group
   6. Monitoring group and individual progress

General Advice to the Instructor

Experience in calculus reform suggests that this new approach involves increased demands on the instructor, particularly in the first term of implementation. The time required for preparation of each class meeting, creation of materials, assisting students with outside projects, incorporating technology, and the varied forms of assessment will be greater than in a traditional course. However, experience gained the first time through will make it much easier in subsequent terms. These materials, as well as the extensive resources now available on collaborative learning and reformed calculus, can be tapped to reduce your workload. You will find the rewards, in terms of increased student involvement and enthusiasm, are considerable. Following are suggestions you may wish to incorporate to encourage active learning in your classroom.

Group-work

Many of your students will have previous experience in collaborative learning but others may not. Depending upon your class you may need to devote twenty minutes or so to a discussion of "What Makes for an Effective Group?" (See included student hand-out).

The size of your class and the type of furniture or desks are major considerations for the instructor. In a large auditorium with fixed seats, groups of two may work best.

In most cases the instructor should experiment with:

a) size of the groups-- Four is often used, but some research suggests a size of three may be more effective.

b) selection method-- Some instructors favor a "Johnson model" (one strong student, two of medium ability, and one weak student in every group.) Others believe that at a minimum, each group should include one strong student. In addition, it is believed by some that gender and ethnic balance should be achieved. Another view is that random selection such as counting off numbers or drawing from a hat works quite well. For the in-class projects, the instructor can create the groups, whereas, for the out-of-class projects the logistics may work better if the students choose their own groups. Some students have work schedules or other commitments at home.

c) types of group projects-- It's probably best to start with something short, where the students are sure to experience success, and then move on to more challenging problems. In any case, the task must be clearly defined, the purpose of the activity must be given explicitly, and the end product with the method of evaluation for the activity should be clearly specified.

Student Presentations

The instructor should use both spontaneous presentations of short problems (perhaps offer a "bonus point" for board work) as well as more formal, prepared presentations that follow a group-work activity. Presentations are more beneficial to the entire class if the presenter has ample time to prepare. It may be helpful if the presenter could be given a transparency and an overhead pen in advance to prepare the written solution ahead of time.

Student presentations will work better if you have a class discussion on effective presentations early in the semester and if each student completes an evaluation form to be given to the presenter (see sample form).

Time Consideration

The instructor should not be concerned about "covering" everything, rather it is more important for students to have meaningful experiences that will lead to understanding of concepts and long-lasting learning.

The group-work and student presentations take time, usually more than you anticipate. It will be helpful to give the assignment or project in advance so the students could work on it or at least read it to determine what must be accomplished. Announcing time limits for specific parts of an activity ("Fifteen minutes to complete...") or for student presentations may help to maximize use of class time. For long-range projects, you may want to give intermediate deadlines for rough drafts or specific parts of the project. Students tend to procrastinate.

Graphing Calculators

Instructors should allow the students to learn as they progress in the course. It's not useful to take several class days to give a course in how to operate the calculator. This time will be needed for the activities. Introduce new techniques only as you need them. Many of the students will have some experience with the graphing calculator. Just allow other students to learn from them.

On a test you may want to have a Part I (no calculators allowed) and a Part II (calculators allowed). For this it may be helpful to use colored paper for say, Part I. Students must turn in Part I before they pick up Part II.

Assessment of the Activities

The instructor should determine how these activities could be linked to a grading scheme in advance. Students always want to know "How much does this count?". In addition, for a group project, you may wish to ask each member to indicate the percentage of work done by each member (making sure the percentages total 100). This will allow you to develop a "multiplier" to affect the group grade.

Advice to the Student:

GUIDELINES FOR EFFECTIVE GROUP WORK IN MATHEMATICS

- Move into your groups quickly, sit close together, use first names, and get right to work.

- Don't engage in "off-task" discussion.

- Make it your responsibility to encourage everyone to participate.

- Read the instructions and the given information for the assignment aloud at the beginning of the session. If there are several parts, read the instructions for each part aloud just before beginning to work on that part. The idea is to get all of the facts into the "record" so that you are sure that everyone is using the same information. This is particularly important when only one copy of the assignment is given to the group.

- Plan and monitor the group's progress toward completion of the assignment. Be aware of time constraints, and keep track of what you have accomplished and what you are currently doing. It is important, and appropriate, to ask each other how what you are doing will help you complete the assignment.

- Listen carefully to each other without interrupting. Respond to, or at least acknowledge, comments made and questions asked by other group members. If you don't understand the information that someone is presenting, try to paraphrase what was said, or ask someone to help you paraphrase it.

- If someone uses a word in a way that you don't understand, ask them what it means. Making sure that everyone is using a technical term correctly is essential to successful communication in mathematics.

- Challenge unjustified assertions, even if they sound right. Ask the person who made the statement to explain his or her reasoning. Be aware that it is possible to get the right answer for the wrong reasons. If you just assume that everyone understands the reasons for assertions, you might be missing important information that could help you solve the problem.

- Take every opportunity to ask questions. This can take a variety of forms, and is perhaps the most important activity in mathematical discussions. (See the reverse side for more on this.)

This course includes group work because experience suggests that talking about mathematics is an important part of learning mathematics for many people. Take full advantage of this opportunity by making a genuine effort to construct the best explanations you can, and help others succeed by asking questions that motivate them to give their best explanations as well.

WHY ASK WHY?

To fully grasp new concepts in mathematics, you must understand how the new ideas relate to those previously learned. One way to ensure that new information is presented to you in context and at the appropriate rate is to ask questions. Asking questions is also a basic tool of mathematical research. Even simple questions about a situation can lead to interesting results. Take the responsibility for asking questions about situations you don't understand or about those you think that your group should explore more deeply.

Here are some suggestions for encouraging question-asking in your group.

1. Do not evaluate or criticize a question. Respond to it cheerfully.
2. Express appreciation for anyone who asks a question. (e.g. "That's a good question." or "I'm glad you asked me that."
3. Ask questions yourself. Be a model for the entire group. (e.g. "Here's something I don't understand. Does anyone know why...")
4. If you see that others are lost, give them an opportunity to ask a question by reviewing your work. (e.g. "let's check to see that we all understand what we've done so far...")
5. Try explaining something by asking a series of simple questions. (Teachers do this routinely because it appears to be more effective than just telling someone the answer.)
6. Make a habit of explaining your reasoning and ask others to do the same. (This provides even more opportunities to ask and answer questions as group members compare their conceptions of the problem.)
7. Come to class prepared. This doesn't mean knowing all the answers. It means knowing what questions you have.
8. Approach the group work with the view that you are responsible for
   a) helping everyone else to learn,
   b) asking interesting questions, and
   c) making the experience an enjoyable one.

Learning is enjoyable. Remember what it was like when you were three years old? If you don't, observe a young child exploring his or her environment. Learning can be just as enjoyable for you.

This page is based on a discussion in "Cooperative Learning Using a Small Group Laboratory Approach", by Julian Weissglass, Chapter 10 in the book Cooperative Learning in Mathematics: A Handbook for Teachers, Neil Davidson, ed., Addison-Wesley Publishing Company, 1990.

Brief Calculus Activity: **Visualizing a Function from a Verbal Description**

Purpose: Students will come to see that quantities possess a functional relationship in a wide variety of real world contexts and to express these relationships graphically.

Background Needed: Students have been exposed to the notion of a function in previous algebra courses.

Course Goals Addressed:

- will demonstrate the ability to communicate effectively in mathematics to various audiences
- will develop the ability to use calculus concepts to interpret quantitative information qualitatively
- will use graphs to help conceptualize and solve problems

Criteria for evaluation:

Each student will demonstrate:

1) understanding of the statement of each problem
2) the logical steps needed for solution
3) appropriate use of standard English and mathematical notation
4) clear and reasonably accurate visual representations of the graphs
5) a coherent explanation

Instructions:

1. Match the stories with three of the graphs in Figure 1, and write a story for the remaining graph. (Harvard calculus project)

(a) I had just left home when I realized I had forgotten my books and so I went back to pick them up.
(b) Things started out fine until I had a flat tire.
(c) I started out calmly, but sped up when I realized I was going to be late.

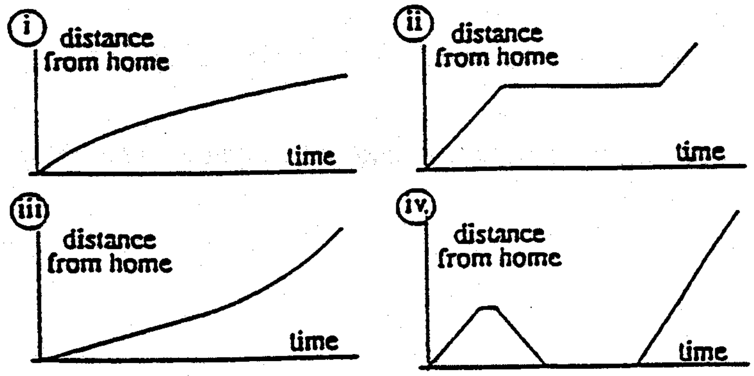


Figure 1

2. A young person is playing with a yo-yo which has a weak string. After the yo-yo goes up and down five times the string breaks. Graph the height of the yo-yo as a function of time.

3. Generally, the more fertilizer that is used, the better the yield of the crop. However, if too much fertilizer is applied, the crops become poisoned and the yield goes down rapidly. Sketch a possible graph showing the yield of the crop as a function of the amount of the fertilizer applied.

4. A runner jogs slowly for five minutes to warm up. She then runs hard for 20 minutes at which time she meets a slower runner and they run together for 10 minutes. She then runs hard for another five minutes and quits. Graph her speed as a function of time.

5. For each of the following four scenarios identify the two quantities that vary, and decide which should be represented by an independent variable and which by a dependent variable. Graph the relationship and write a sentence or two to explain the shape of your graph. (Project CALC from Duke)

(a) The amount of money earned on a part-time job and the number of hours worked.

(b) The cost to make similar pizzas of different diameters.

(c) The postage on a first class letter and the weight of the letter.

(d) The number of cans of beer consumed by a fraternity and the day of the week.

6. Problem 6 is about supply and demand curves. Economists are interested in how the quantity of a good which is manufactured and sold, q, depends on its price, p. They think of quantity as a function of price. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. There are two graphs in Figure 2 relating p and q: the supply curve which represents manufacturer's behavior and the demand curve which represents consumer behavior.

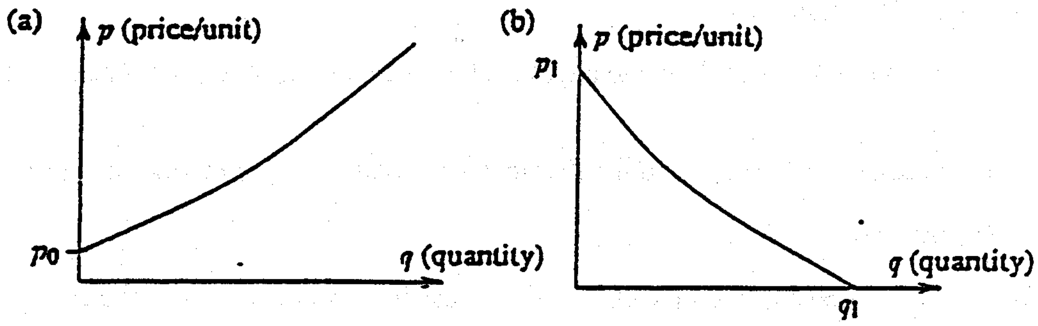


Figure 2

a) One of the graphs in Figure 2 is a *supply curve* and the other is a *demand curve.* Which is which? Why?

b) The price p_0 in Figure 2 represents the price below which the manufacturers are unwilling to produce any of the good. What do the price p_1 and the quantity q_1 in Figure 2(b) represent in practical economic terms?

7. In her *Guide to Excruciatingly Correct Behavior,* Miss Manners states:

There are three possible parts to a date, of which at least two must be offered: entertainment, food and affection. It is customary to begin a series of dates with a great deal of entertainment and the merest suggestion of affection. As the amount of affection increases, the entertainment can be reduced proportionately. When the affection has replaced the entertainment, we no longer call it dating. Under no circumstances can the food be omitted.

Based on this statement, sketch a graph showing entertainment as a function of affection, assuming the food to be constant. Mark the point at which the relationship starts, as well as the point at which the relationship ceases to be called dating.(Harvard project)

To the Instructor:

Hints and Comments:

Students will work in pairs to answer the questions. After 45-50 minutes, four of the pairs will be asked to prepare a transparency with their solution to one of the following problems: #2, #3, #4, or #5. The selected pairs will present and defend their solutions before the entire group. In addition, each student will turn in their completed solutions. Instructor will grade worksheets and discuss class presentation.

This activity may take more than one day, thus the instructor may assign part as homework.

The self-assessment is a suggested list that may be modified as the instructor pleases.

Suggested Follow Up:

- At the end of the presentations the class will discuss the solution to one of the following problems: #2, #3, #4, or #5.
- At the end of the presentation the instructor will give a summary of the way to approach and work this problem.
- The instructor should evaluate the group's work on the other problems and individual participation in the group.

Brief Calculus Activity: **Functions in a Real World**

Purpose: The purpose is to give students a deeper understanding of functions that model real world situations, especially those that will arise in optimization problems.

Background Needed: The student understands the concept of a function and has had experience giving rules for simple functions based on verbal descriptions in an earlier Brief Calculus activity. In addition, the student knows how to produce the graph of a given function using a graphing utility.

Course Goals Addressed:

- will develop mathematical modeling ability to solve real world problems
- will demonstrate the ability to communicate effectively in mathematics to various audiences
- will use concepts and techniques of calculus to analyze functional relationships
- will develop the ability to use calculus concepts to interpret quantitative information qualitatively
- will use graphs to help conceptualize and solve problems

Criteria for Evaluation:

Each student will demonstrate:

1) full participation with the group
2) understanding of the statement of each problem
3) understanding of the logical development needed to create the functional model appropriate to each situation
4) accurate use of mathematical notation
5) a clear and accurate visual representation of the graph
6) a coherent explanation

Instructions: Read each problem situation and answer the related questions. You will be asked to write a function which expresses the relationship between the variables in the problem.

1. Your company produces a product for which the variable cost is $15.00 per unit and the fixed costs are $400. The product sells for $25.00. Let x be the number of units produced.

(a) Write the total cost C as a function of the number of units produced.

Note: total cost = C(x) = variable costs + fixed costs

C(x) = (number of units)(variable cost per unit) + fixed costs

C(x) = ____

(b) Write the revenue R as a function of the number of units produced

R(x) = ____

(c) Write the profit P as a function of the number of units produced

P(x) = ____

(d) Using a graphing utility, graph the cost function C(x) and the revenue function R(x).

Find the point of intersection.________________
This point would be the break-even point, since revenues equal costs.

2. An electronics store needs to order a total of 2400 CD players over the course of a year. It will receive them in several shipments, each containing an equal number of CD players. The shipping costs are $50 for each shipment, plus a yearly fee of $2 for each CD player in a single shipment.

Let x = number of CD players in a single shipment, and write a function for the total yearly shipping costs.

3. A manufacturer produces a certain type of calculator at a cost of $5 each. The current price of this calculator is $15 and sales of the calculator have been averaging 5000 calculators a week. The manufacturer is planning a price increase. From surveys conducted by the marketing department, he knows that for each $1.00 increase in price, weekly demand will decrease by 250 calculators. Write a profit function that describes this situation.

Let x = the number of price increases (of $1.00 each), thus x also equals the number (of 250 calculators) decreases in demand.

Recall that profit = revenue - cost, so

P(x) = (number of items)(price per item) - (number of items)(cost per item)

Think about the new price per item...can you express it in terms of x?

____

Think about the number of items that will be sold after x price increases...can you express that in terms of x? ______________________________

Write the profit function: P(x) =______________________________

Graph the profit function P(x) using a graphing utility. Approximate the number of price increases that will produce maximum profit.__________

Describe what happens if the manufacturer has several more price increases than the number you calculated above.

4. A container manufacturing company has been contracted to manufacture open-top rectangular storage bins for small automobile parts. They have been supplied with 20 cm x 24 cm sheets of tin. The bins are to be made by cutting squares of the same size from each corner of a sheet and then bending up the tabs and spot-welding the corners. The figure below shows two diagrams of the sheet of tin with the corner pieces removed and the open box formed.

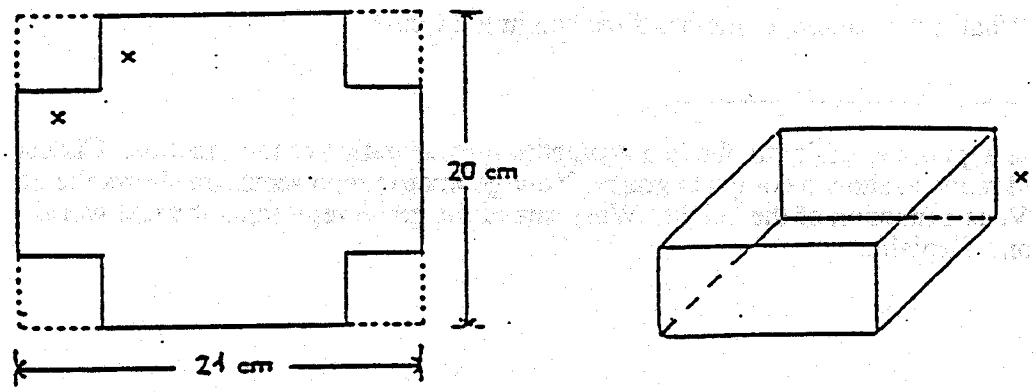


Figure for Problem 4

Let x = the side length of the square to be cut from each corner. Express the dimensions of the bin in terms of x.

height = ____________________________

width = ____________________________

length = ____________________________

Write the volume of the bin as a function of the height:

V(x) = ________________________

Find some specific values:

(a) What is the volume of the bin if the height is 2 cm?

V(2) = ________________________

(b) What is the volume of the bin if the height is 8 cm?

V(8) = ________________________

(c) What is the volume of the bin if the height is 11 cm?

V(11) = ________________________

Next, use a graphing utility to obtain a geometric representation of the function. Choose your domain and range to show a complete graph. Your geometric representation shows the graph of the volume, V, as a function of the height. What part of the graph represents the real world physical situation? Explain.

Redraw the graph, using an appropriate domain and range to show this portion of the graph. Notice the highest point of the graph.

What is the value of the x-coordinate at this maximum point?____________

What is the value of the y-coordinate? ________________

What is the maximum volume that the bin described in the problem can have?_____________

Give a rough sketch of the graph from your calculator or computer screen.

5. An island is at point A, 6 miles off shore from the nearest point B on a straight beach. A store is at point C, 7 miles down the beach from B.

(a) Draw a diagram showing the relative positions of A, B, C, the island and the store.

(b) A man is on the island and he wishes to get to the store. He can row at the rate of 4 miles per hour and walk at the rate of 5 miles per hour. Should he row to B and walk to C? Should he row directly to C?

Assume he lands at some point, let's call it P, located between B and C. Let x = the distance between B and P. Add P and x to your diagram. What are the possible values for x? What makes sense for x?

(c) Write a function for the total time (time rowing + time walking) that the trip will take. Recall that time = distance/rate.

T(x) = ____

To the Instructor:

Hints and Comments:

The students will divide into groups of four. Each student will receive a worksheet to be completed and turned in by the end of class. Worksheet problems will be discussed by the group. Toward the end of class a volunteer will be recruited to present the solution to Problem #4 using a calculator projection device, if available, or a transparency otherwise. Instructor will grade worksheets and discuss class presentation.

This activity requires a graphics calculator.

Problem number three is designed to use a graphing calculator to visualize the shape of the graph. We recommend the profit function be entered into the calculator without algebraic simplification. By using the calculator students should be able to find the number of price increases that maximize profit. In addition, by using the graph students should see how graphic representation of data aids in decision making.

Problem number four is designed to aid the students in setting up functions to which techniques of calculus can be applied.

Suggested Follow Up:

- At the end of the presentations the class will discuss the solution to problem #4.
- At the end of the presentation the instructor will give a summary of the way to approach and work this problem.
- The instructor should evaluate the group's work on the other problems and individual participation in the group.

Brief Calculus Activity: **Discovering the Differentiation Rules, Part I**

Purpose: The student will discover some of the rules of differentiation and will gain a deeper understanding of the limit process by applying the definition of the derivative.

Background Needed: Students will have basic algebra skills, understand functional notation, understand limit notation, and can use a graphing calculator.

Course Goals Addressed:

- will demonstrate the ability to communicate effectively in mathematics to various audiences
- will use concepts and techniques of calculus to analyze functional relationships
- will demonstrate the mathematical manipulative skills necessary for problem solving

Criteria for Evaluation:

1. works accurately and obtains correct answers
2. uses English correctly
3. writes mathematical notation correctly
4. demonstrates understanding with clear and coherent explanations
5. exhibits appropriate logical development

Activity #1 (1st class period)

Graphing Calculator Exercise:

On a graphing calculator graph $y_1 = 1/x$ and $y_2 = -x+2$. Your graph should look similar to figure 1. Use the tracer to keep the graph centered close to an x-value of 1. Use the zoom-in feature several times on the graph. Describe what happens to the graph of $y_1 = 1/x$ as you zoom-in several times.

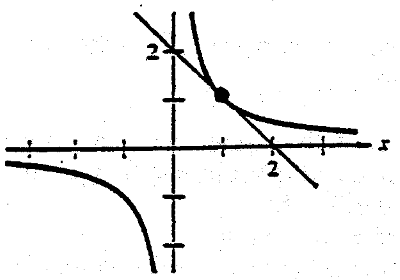


Figure 1

The line $y_2 = -x+2$ is a tangent line to the graph of $y_1 = 1/x$ at the point (1,1). The word "tangent" comes from a Greek word which means to "touch". Notice that the graph turns away from the tangent line. While the tangent line touches the graph at a point, it may cross the graph at some other point.

Exercise: For each of the following graphs draw the tangent line at the point indicated.

1. 2. 3.

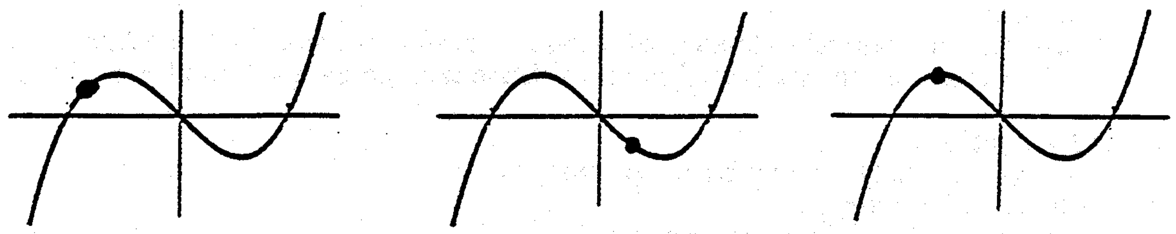


Figure 2

From the graphs shown above say whether the tangent line is the "same" or "different" at various points on the graph. Discuss briefly.

Recall that to find the <u>equation</u> of a line you need to know, a point on the line, (x_1,y_1), and the slope of the line, m. We use the formula $y - y_1 = m(x - x_1)$ and algebra to find the equation of the line.

Consider the graph in figure 1, if the point shown is (1,1). What else would have been needed to determine the equation of the line shown? ________.

For various applications we generally know the equation and a point on the graph however we need to be able to determine the slope of the tangent line at that point. We use the tangent line to:

1) aid in graphing the equation of the function
2) locate relative maximum and minimum points
3) approximate functional values "close" to the point of tangency

You can determine the slope of the tangent line at the point on the graph of $y = f(x)$ by the following reasoning. Consider the following graph in figure 3.

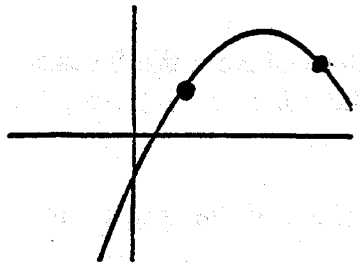


Figure 3

Start at a point x, the y-value of the point is f(x).
Move to the right "h" units, you are at (x+h) the y-value is __________.
Draw a tangent line at (x,f(x)).
Draw a line through the two points (x,f(x)) and (x+h,f(x+h)). This line is called a secant line.

The slope of the secant line is m_{secant} = __________.

If h gets smaller and smaller ($h \to 0$) compare what happens to the slope of the secant line to the slope of the tangent line. (**Hint** draw several secant lines between x and x+h).

This reasoning leads us to conclude that $m_{tangent} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

The notation we use is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. f'(x) can often be found using algebra.

Example:
Find the derivative of $f(x) = x^2 - 2x + 3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - 2(x+h) + 3] - [x^2 - 2x + 3]}{h}$$ **substitute f(x+h) and f(x) in f'(x)**

$$= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2 - 2x - 2h + 3] - [x^2 - 2x + 3]}{h}$$ **multiplication**

$$= \lim_{h \to 0} \frac{x^2 + 2xh - h^2 - 2x - 2h + 3 - x^2 + 2x - 3}{h}$$ **removal of ( ) and cancel**

$$= \lim_{h \to 0} \frac{2xh - h^2 - 2h}{h}$$ **simplify**

$$= \lim_{h \to 0} \frac{h(2x - h - 2)}{h}$$ **cancel out h**

$$= 2x - 2$$ **determine limit**

Notice that the function f'(x) = 2x - 2 was derived from the function f(x) = x^2 - 2x + 3, hence the name "derivative." Substitution of a value for x in f'(x) gives the slope of the tangent line at the point with that x-value.

Example f'(3) = 2(3) - 2 = 4. Thus 4 is the slope of the tangent line at the point (3,6).

Exercise: Find the derivative of x^2 + 3x - 2.

Find the slope of the tangent line at x = -1.

This exercise shows how much work is involved in determining the derivative using the definition of the derivative as a limit. Mathematicians have developed techniques which greatly reduce the work needed to determine the derivative. These techniques are introduced beginning in activity #2.

To the Instructor:

Hints and Comments:

This activity is designed to present to the student the formal definition of the derivative. The next activity is the discovery of the power rule. Once the students have seen the definition of the derivative computation of the derivatives is best accomplished by other techniques.

Suggested Follow Up:

- This exercise should be done by individual students.
- The instructor may grade this exercise at his or her own discretion.
- This is the beginning of those techniques of differentiation that students need to approach real world problems.

Brief Calculus Activity: **Discovering the Differentiation Rules, Part II**

Purpose: The student will discover some of the rules of differentiation and will gain a deeper understanding of the limit process by applying the definition of the derivative.

Background Needed: The student will know function notation and basic algebra skills.

Course Goals Addressed:

- will demonstrate the ability to communicate effectively in mathematics to various audiences
- will use concepts and techniques of calculus to analyze functional relationships
- will demonstrate the mathematical manipulative skills necessary for problem solving

Criteria for Evaluation:

Each student will demonstrate:

1. computational accuracy
2. use of proper English
3. writing mathematical notation correctly
4. understanding with clear and coherent explanations
5. appropriate logical development

Instructions: (2nd class period)

Determine f'(x) for each of the following functions using the definition of the derivative, where f'(x) is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$ provided the limit exists.

Note that the following binomial expansions will be helpful:

$$(x + h)^2 = x^2 + 2xh + h^2$$

$$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^2 + h^4$$

$$(x + h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

Determine f'(x) for each of the following:

Example: $f(x) = x^2$ f'(x)= 2x

1. $f(x) = x^3$ f'(x) = __________

2. $f(x) = x^4$ f'(x) = __________

3. $f(x) = x^5$ f'(x) = __________

Questions:

1. When f'(x) was determined only the second term in the binomial expansion of (x+h) was not eliminated. Explain why the other terms were eliminated.

2. Compare f(x) and f'(x) in each of the four cases. What is the relation between the power of x in f(x) and the coefficient of x in f'(x)?

3. What is the relation between the power of x in f'(x) and the power of x in f(x)?

By using the relationships observed above, determine the derivative of:

1. $f(x) = x^6$ f'(x) = __________
2. $f(x) = x^7$ f'(x) = __________
3. $f(x) = x^{20}$ f'(x) = __________
4. $f(x) = x^{100}$ f'(x) = __________

By reasoning as above, and/or by consulting a textbook, find the formula for differentiating.

$f(x) = x^n$ $\qquad$ $f'(x) =$ ______________

This formula is called the power rule and it holds for all real numbers, not just for the positive integers. Since the power rule is true for all real numbers n, the following exercises will attest to its usefulness.

Exercises: Find the derivative of each of the following functions using the power rule.

1. $f(x) = x^8$ $\qquad$ $f'(x) =$ _________
2. $f(x) = \frac{1}{x^2}$ $\qquad$ $f'(x) =$ _________
3. $f(x) = x^{.5}$ $\qquad$ $f'(x) =$ _________
4. $f(x) = x^{0.11}$ $\qquad$ $f'(x) =$ _________

To the Instructor:

Hints and Comments:

The class will form small groups during class time and also work individually outside of class to solve these assessment activities.

This activity is designed to be presented after the student has been exposed to the formal definition of the derivative. It is the discovery of the power rule.
Once the students have discovered the power rule, time restraints seem to make the presentation of the other differentiation rules in a different mode advisable.

Suggested Follow Up:

- This exercise should be done by individual students.
- The instructor may grade this exercise at his or her own discretion.
- This is the beginning of those techniques of differentiation that students need to approach real world problems.

Brief Calculus Activity: **Discovering the Differentiation Rules, Part III**

Purpose: The student will use some of the techniques of differentiation to gain the ability to use the derivative to solve applied problems.

Background Needed: Students will have basic algebra skills, understand functional notation, limit definition of a derivative, and knowledge of the power rule.

Course Goals Addressed:

- will demonstrate the ability to communicate effectively in mathematics to various audiences
- will use concepts and techniques of calculus to analyze functional relationships
- will demonstrate the mathematical manipulative skills necessary for problem solving

Criteria for Evaluation:

1. works accurately and obtains correct answers
2. uses English correctly
3. writes mathematical notation correctly
4. demonstrates understanding with clear and coherent explanations
5. exhibits appropriate logical development

Activity #3 (3rd class period)

Notation: The calculus was developed independently in the late 1600s by Sir Isaac Newton in England and Gottfried Wilhelm Leibniz in Germany. The result of this independent development is diversity of notation. There are several ways of asking you to find the derivative.

If $f(x) = x^2 - 2x + 3$ $f'(x) =$ ______________

If $y = x^2 - 2x + 3$ $y' =$ ___________

If $y = x^2 - 2x + 3$ $\frac{dy}{dx} =$ ___________

If $y = x^2 - 2x + 3$ $\frac{d(x^2 - 2x + 3)}{dx} =$ _______________

The techniques presented here may be represented in your textbook in a variety of ways. The important ability for you to develop is the ability to look at a problem and determine which technique needs to be applied to come up with the derivative. We wish to simplify the notation which will make remembering the techniques easier. To do this functions will be represented by the letters u and v instead of f(x) and g(x). Derivatives of u and v are represented by u' and v'.

At this point you should be able to determine the derivative of a function using the power rule.

If $y = x^n$ then $y' = nx^{n-1}$

Example 1: $y = x^3$ $y' = 3x^2$

Constant Times a Function Rule:

Consider the following problem if $y = 4x^3$, find y'.

Notice the difference between this problem and example 1. The difference is the function $f(x) = x^3$ is now multiplied by the constant 4. To handle problems such as this we use the constant (denoted by c) times a function technique.

If $y = c \cdot u$ then $y' = c \cdot u'$.

Thus if $y = 4x^3$ then $y' = 4 \cdot 3x^2 = 12x^2$

Example 2: $y = 5x^4$ $y' = 20x^3$

Exercise:

a) $y = 5x^{10}$ $y' =$ __________

b) $y = 8x^{.5}$ $y' =$ __________

c) $y = 6x^5$ $\frac{d(6x^5)}{dx} =$

Sum and Difference Rule:

Consider the following problem if $y = x^3 - x^2$, find y'. The difference between this problem and previous examples is that the functions $v = x^2$ is subtracted from the function $u = x^3$. To handle problems such as this we use the sum and/or difference technique.

If $y = u \pm v$ then $y' = u' \pm v'$

Example 3: $y = x^3 - x^2$ $y' = 3x^2 - 2x$

Note: The derivative of a constant is 0.

Recall that $x^o = 1$ thus if $y = 5 = 5 \cdot x^0$ $y' = 5 \cdot 0 \cdot x^{-1} = 0$

Example 4: $y = 3x^2 - 2x + 4$ $y' = 6x - 2$

Exercise: a) $y = 7x^4 - 3x^3 + 4x^2 - 8x + 13$ $y' =$

b) $y = 8x^{.5} + 9x^{0.25} - 5$ $y' =$

Product Rule:

There are some instances where you need to take the derivative of the product of two functions, i.e. $y = u \cdot v$ where u and v are two functions. If $y = (3x-2)(2x-5)$ then u would be 3x-2 and v would be 2x-5. The derivative of the product of two functions is not the product of the derivatives. You should verify that for $y = (3x-2)(2x-5)$ that $y' \neq u' \cdot v'$. The technique for finding the derivative of the product of two functions is

If $y = u \cdot v$ **then** $y' = u' \cdot v + u \cdot v'$

The product rule is given in a form that is easier to remember. Notice the terms in $u' \cdot v + u \cdot v'$ correspond to the order of the original problem $y = u \cdot v$. Until you gain experience with the product rule determine u, u', v, v,' and substitute these values into the formula for y'.

Example 5: $y = (3x-2)(2x-5)$

$u = 3x-2$ $u' = 3$
$v = 2x-5$ $v' = 2$

$y' = (3)(2x-5) + (3x-2)(2)$

y' often should be simplified. If the sum of these products is not easily simplified, a graphing calculator can be used to work with the derivative.

Exercise: a) $y = (4x-6)(3x+1)$ $y' =$

b) $y = (2x-1)(4x-1)$ $y' =$

Quotient Rule:

There are instances where it is necessary to take the derivative of the quotient of two functions. The technique is

If $y = \frac{u}{v}$ **then** $y' = \frac{u' \cdot v - u \cdot v'}{v^2}$

Again this technique is presented in a form that is easy to remember. The numerator is very similar to the product rule. The denominator is v squared. Note: the values of u and v come from the original problem.

Example 6: $y = \frac{3x-1}{2x+4}$

$u = 3x-1 \qquad u' = 3$
$v = 2x+4 \qquad v' = 2$

$$y' = \frac{(3)(2x+4) - (2x+4)(2)}{(2x+4)^2}$$

y' often should be simplified. You should be very careful with the - sign, since the multiplication should be done before subtraction.

Exercise: a) $y = \frac{3}{7x^5}$

b) $y = \frac{3x-5}{7}$

Generalized Power Rule:

There are some applied problems where we need to work with a function raised to a power. The technique used is similar to the power rule. The generalized power rule is:

If $y = u^n$ then $y' = n \cdot u^{n-1} \cdot u'$

Example 7: $y = (3x-2)^5$ notice $u = 3x-2$ and $u' = 3$

$y' = 5(3x-2)^4 \cdot 3$
$= 15(3x-2)^4$

Exercise: a) $y = (2x-1)^7 \qquad y' =$

b) $y = (x^2 + 3)^5 \qquad y' =$

More complicated problems can often be solved by using more than one technique. For example if $y = \frac{2(3x-5)^2}{(2x-1)}$, you have

$y = \frac{u}{v}$ where $u = 2(3x-5)^2$ and $v = (2x-1)$. To determine u' you need to use the generalized power rule.

To the Instructor:

Hints and Comments:

This activity is designed to introduce the students to the techniques of differentiation needed to solve business calculus problems. The exercises should be supplemented from the text or some other sources.

Suggested Follow Up:

- This exercise should be done by individual students.
- The instructor may grade this exercise at his or her own discretion.

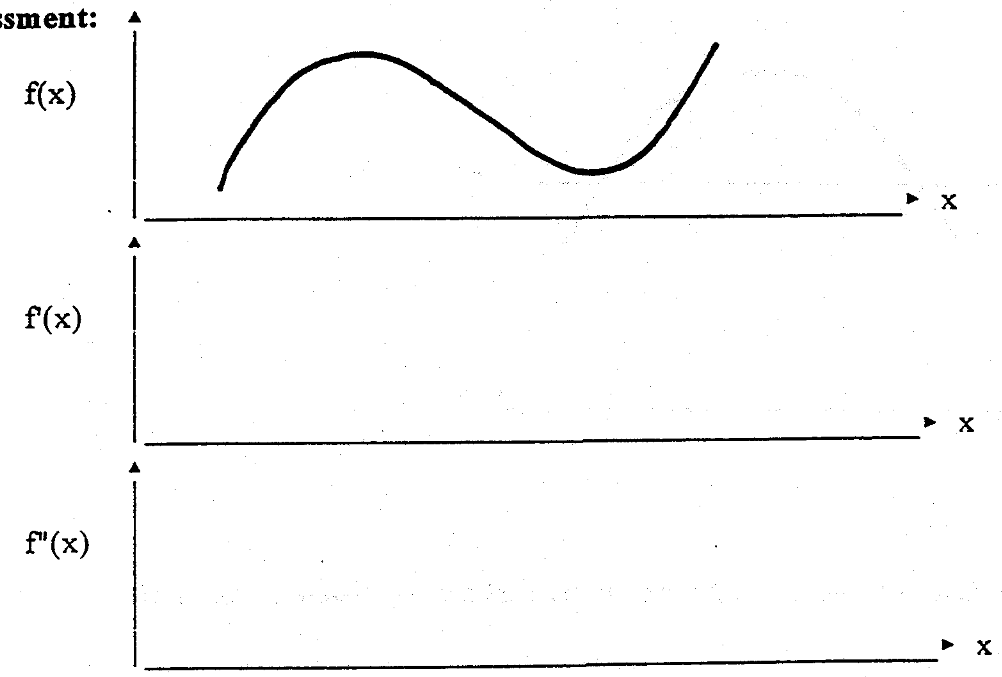
Brief Calculus Activity: **Connections between Graphs: Functions and Derivatives**

Purpose: The student will discover relations between a function and its' first and second derivatives.

Assumptions: Each student has learned how to calculate derivatives and understands its interpretation as the slope of the tangent line to the graph of the function at a point.

Instructions: The graph of a function $f(x)$ is shown below, from this graph, use the idea of slope to sketch the first derivative in the space below $f(x)$. Next use $f'(x)$ as the original function and sketch $f''(x)$ using only the idea of slopes of tangent lines. As a group discuss your sketches and (agree?) that all sketches of the curves should look alike. Then answer the questions below to test your sketches.

Assessment:



1. Does your sketch of $f'(x)$ cross the axis where $f(x)$ has a relative maximum and minimum? ___________

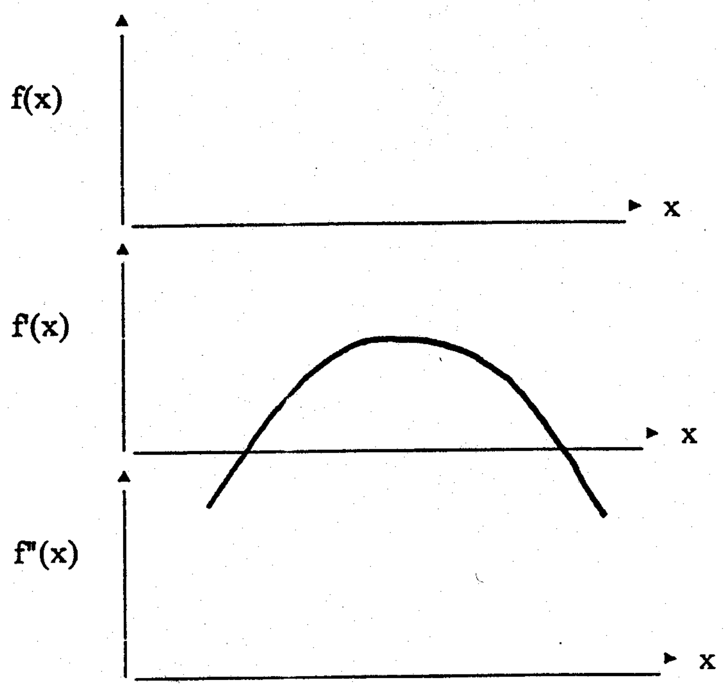
2. Is your sketch of $f'(x)$ look like a u-shape? __________

3. $f'(x)$ should have a relative minimum. Have you graphed $f''(x)$ crossing the axis at this point? __________

4. Does your sketch of $f''(x)$ look almost like a straight line? __________

If your answers to these four questions are yes, you should have the correct sketches.

Now consider the sketch of f'(x) below. As a group discuss what the sketch of f(x) and f"(x) should look like. Sketch in f"(x) and then f(x). Test your sketches by answering the questions below.



1. Does your sketch of f"(x) cross the axis where f'(x) has its relative maximum? ___________

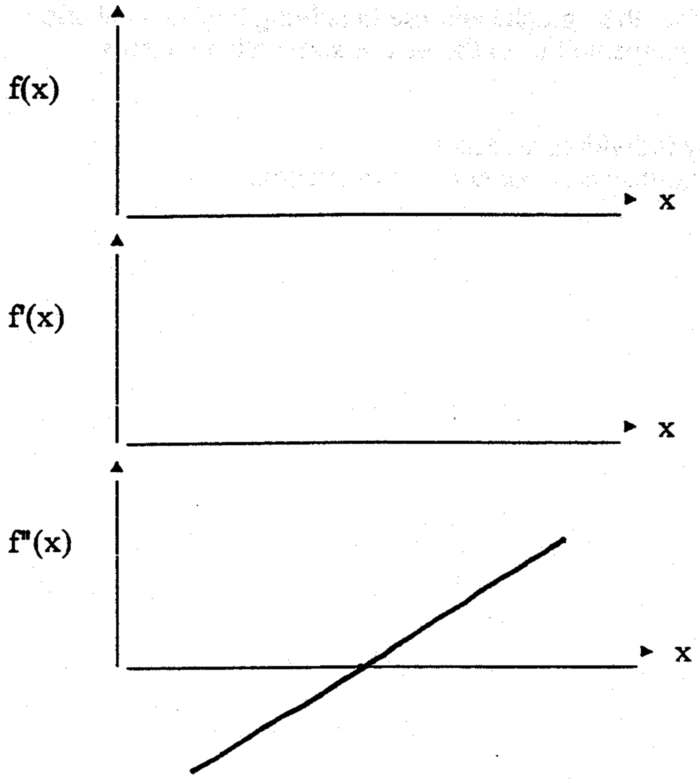
2. Does f(x) have an inflection point where f"(x) crosses the axis? __________

3. Does f(x) have a minimum and a maximum where f'(x) crosses the axis? __________

4. Is your sketch of f(x) concave down where f"(x) is below the axis? __________

If your answers to these four questions are yes, you should have the correct sketches.

Now consider the sketch of f"(x) below and look back at your first sketches. As a group discuss how to get several different sketches of f(x) directly from f"(x). Sketch both sketches of f(x) and answer the following questions:



1. Does one of your sketches of f(x) look like the first exercise? __________

2. Does your other sketch have no relative maximum or minimum? __________

3. Are both sketches of f(x) concave up where f"(x) is above the axis? __________

If the answers to these three questions are yes, then you should have the correct sketches.

Note that we did not try to sketch f'(x). Discuss this. One of the main points in our lesson today was that f"(x) = 0 only suggests that there <u>may</u> be an inflection point on the graph of f(x). It does not imply a RELATIVE MAXIMUM OR MINIMUM.

To the Instructor:

Hints and Comments:

This activity is designed to introduce the students to the idea of extrema. The student will learn about the first and second derivative, their grapha and use in solving business calculus problems. The exercises should be supplemented from the text or some other sources.

Suggested Follow Up:

- This exercise should be done by individual students.
- The instructor may grade this exercise at his or her own discretion.

OPTIMIZATION

The next four course activities are designed to help the student make the transition from the theory of finding the derivative of a function to the concrete use of the derivative to optimize a function derived from a real world problem. The student will discover how to find the extrema of a function, determine the concavity of a function, and apply this knowledge to the optimum solution to a given real world problem.

The four course activities are:

1. Discovering extrema using the first derivative test.
Time: 20 - 30 minutes
Materials needed: A copy of activity 1 for each student
No extra material needed

2. Identifying extrema using the second derivative test.
Time: 30 - 50 minutes
Materials needed: A copy of activity 2 for each student
No extra material needed

3. Exploring concavity, the second derivative, and inflection points.
Time: 40 - 75 minutes
Materials needed: A copy of activity 3 for each student
No extra material needed

4. Modeling a real world optimization problem.
Time: At least 2 hours
Materials needed: A copy of activity 4 for each student
A box whose ends fold over to make the top and bottom of double thickness
Materials for the students to make audio-visual aids for their presentations

Hints/comments:

The instructor will only succeed through trial and error. The instructor may have to modify these activities to fit his/her own needs.

These activities take up more time than the instructor anticipates.

The box used in activity 4 was originally a box from a moving company. You may use any box but you must change the volume to conform to your box.

Brief Calculus Activity: **Discovering Extrema Using the First Derivative Test**

Purpose: The student will discover that an extrema (extreme value) of a function occurs where the first derivative of a function is zero.

Background Needed: The student has learned how to calculate the derivative of a function and understands its interpretation as the slope of the tangent line to the graph of the function at a point.

Course Goals Addressed:

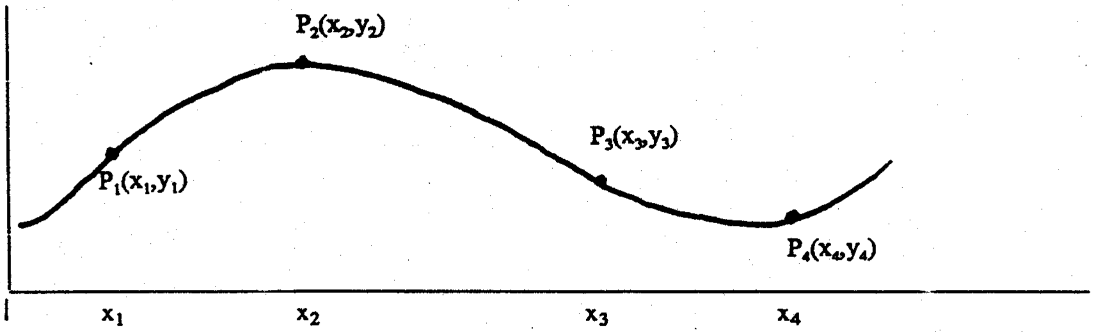
- will communicate clearly in written English and be able to comprehend analyze, and interrogate a variety of written texts
- will communicate orally to be an effective member of the group
- will perform basic mathematical manipulations,display facility with the use of mathematics in framing concepts for mathematical analysis, and interpret data intelligently
- will understand some of the phenomena comprehended by the physical/or life sciences and participate in the experimental process for verifying scientific hypotheses
- will be able to use computers to perform subject-specific tasks appropriate to their major fields

Criteria for Evaluation:

Each student will demonstrate:

1 participation with the group
2 understanding of the statement of problem
3 solutions written in clear, complete sentences
4 accurate use of mathematical notation
5 solutions supported with previously presented concepts with accurate computation

Instructions: A portion of the graph of the function y=f(x) is shown below. From the graph, we want to determine the properties of the derivative at the indicated points. As a group, discuss and formulate your solution to each of the questions. Make sure your solutions are written in clear complete sentences, your answers are supported with concepts previously presented, and that your computations are correct.



1. At each point P_1, P_2, P_3, and P_4, decide if the derivative of the function is positive, negative or zero.

2. What is special about the point P_2 in relation to nearby points?

3. Which other points would you consider special? Why?

4. Write a paragraph describing a method for calculating the points P_2 and P_4 from the formula, y=f(x).(Not involving the use of a graphing calculator.)

5. Apply your method to the function y=f(x) where $f(x)=2x^3-9x^2+12x+1$.

To the Instructor:

Hints and Comments:

At the end of the group exercise, the groups will discuss the results of their investigation and their conclusions. The instructor should present the formal statement of the First Derivative Test for extrema of a function.

Suggested Follow Up:

- The instructor could assign several exercises for group practice in the classroom.
- The instructor should assign a set of five homework problems, from which one can be selected to appear on a short quiz at the beginning of the following class period.

Brief Calculus Activity: **Identifying Extrema Using the Second Derivative Test**

Purpose: The student will discover the possible behaviors of the first and second derivatives at critical values of a function, how the second derivative can be used to help locate extrema, and that f'(x)=0 does not always imply f(x) has an extrema at that point.

Background Needed: The student understands the implications and usage of the first derivative test and can find the second derivative.

Course Goals Addressed:

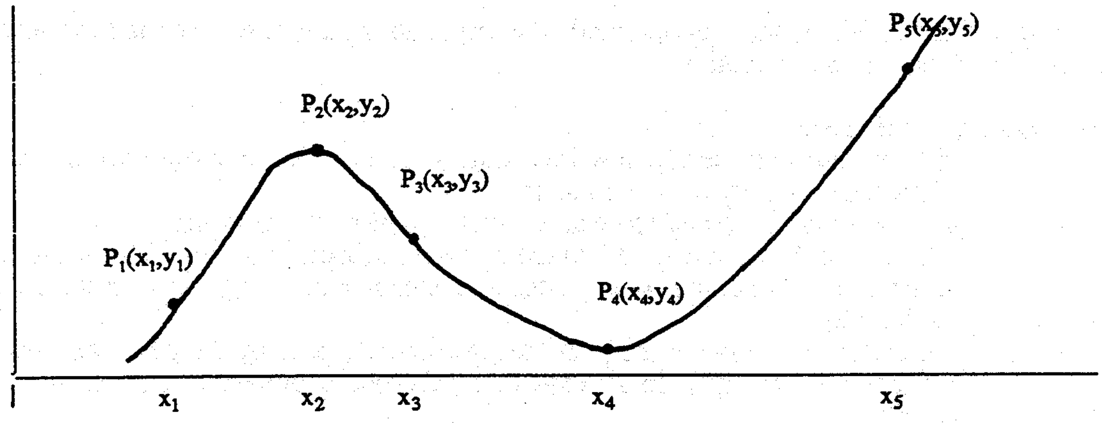
- will communicate clearly in written English and be able to comprehend, analyze and interrogate a variety of written texts
- will communicate orally to be an effective member of the group
- will perform basic mathematical manipulations,display facility with the use of mathematics in framing concepts for mathematical analysis, and interpret data intelligently
- will understand some of the phenomena comprehended by the physical/or life sciences and participate in the experimental process for verifying scientific hypotheses
- will be able to use computers to perform subject-specific tasks appropriate to their major fields

Criteria for Evaluation:

Each student will demonstrate:

1 participation with the group
2 understanding of the statement of problem
3 solutions written in clear, complete sentences
4 accurate use of mathematical notation
5 solutions supported with previously presented concepts with accurate computation

Instructions: A portion of the graph of the function y = f(x) is shown below. From the graph, we want to determine the behavior of the first and second derivatives at the indicated points. As a group, discuss and formulate your solution to each of the questions. Make sure your solutions are written in clear complete sentences, your answers are supported with concepts previously presented, and that your computations are correct.



1. At each labeled point, determine if the first derivative is positive, negative, zero, or does not exist.

2. If the second derivative at P_1 is zero, P_2 is positive, P_3 is zero, P_4 is negative, P_5 does not exist, and P_6 does not exist then what is the relationship between the second derivative and the relative extrema? For points P_2 and P_4? For points P_5 and P_6?

3. Under what circumstances will the second derivative not help find maxima or minima?

4. How can we use the second derivative to help find relative maxima or minima?

5. Use your graphing calculator to graph the function $f(x)=x^3$. Find f '(0) and f "(0). Is there a critical point at x=0? Why won't your assumptions about the second derivative work in this case?

To the Instructor:

Hints and Comments:

At the end of the group exercise, the groups should discuss the results of their investigation and their conclusions. The instructor should present the formal statement of the Second Derivative Test for extrema of a function and the definition of a critical point.

Suggested Follow Up:

- The instructor can assign several exercises for group practice in the classroom, including the function $f(x)=x^4$.
- The instructor could assign a set of homework problems, from which one can be selected to appear on a short quiz at the beginning of the following class period.

Brief Calculus Activity: **Exploring Concavity, Second Derivative, Inflection Points**

Purpose: The student will discover the relationship between concavity and the second derivative and formulate a definition of inflection points.

Background Needed: Each student has learned to take the first and second derivatives of a function, the First and Second Derivative Tests, and the use of the first derivative to determine intervals where the function is increasing or decreasing.

Course Goals Addressed:

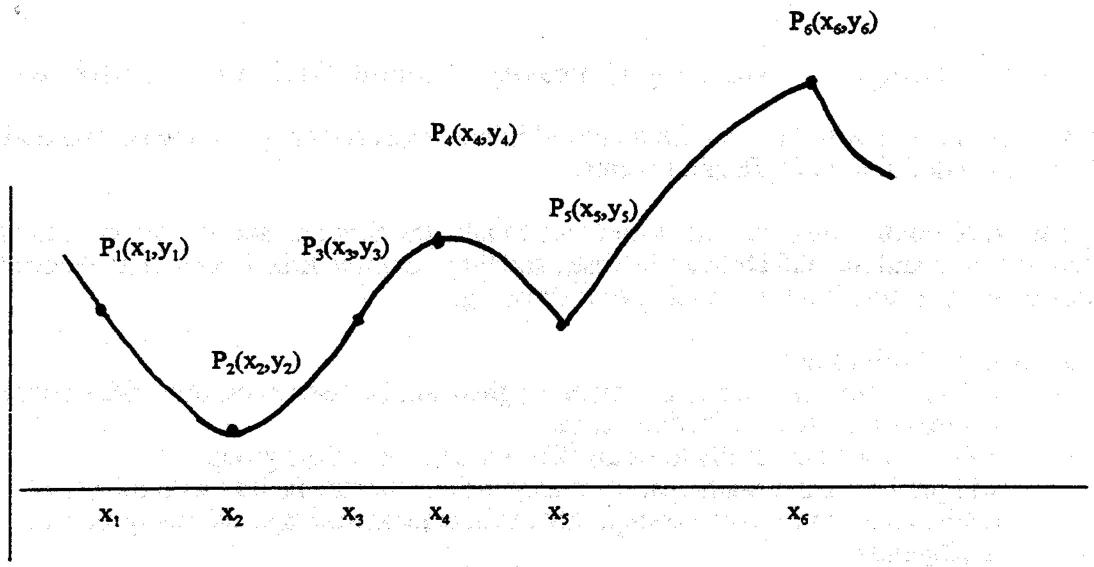
- will communicate clearly in written English and be able to comprehend, analyze and interrogate a variety of written texts
- will communicate orally to be an effective member of the group
- will perform basic mathematical manipulations,display facility with the use of mathematics in framing concepts for mathematical analysis, and interpret data intelligently
- will understand some of the phenomena comprehended by the physical/or life sciences and participate in the experimental process for verifying scientific hypotheses
- will be able to use computers to perform subject-specific tasks appropriate to their major fields

Criteria for Evaluation:

Each student will demonstrate:

1 participation with the group
2 understanding of the statement of problem
3 solutions written in clear, complete sentences
4 accurate use of mathematical notation
5 solutions supported with previously presented concepts with accurate computation

Instructions: A portion of the graph of the function y=f(x) is shown below. From the graph, we want to analyze the way in which the graph changes shape and formulate a definition of an inflection point. As a group, discuss and formulate your solution to each of the questions. Make sure your solutions are written in clear complete sentences, your answers are supported with concepts previously presented, and that your computations are correct.



1. Identify interval(s) on which the slopes of the tangent lines to the graph are increasing.

2. Identify interval(s) on which the slopes of the tangent lines to the graph are decreasing.

3. Where does the change from increasing slope to decreasing slope occur?

4. What can you conclude about the sign of the second derivative on the interval from P_1 to P_5? From P_5 to P_8?

5. How is the sign of the second derivative related to the shape of the graph?

6. What is the value of the second derivative at P_5? This is an inflection point. Formulate a definition of an inflection point.

7. Describe a method for finding inflection points for any function.

8. Test your method on the function $f(x) = x^3 - 3x$. Try it also on $f(x) = x^4$.

To the Instructor:

Hints and Comments:

At the end of the group exercise, the groups will discuss the results of their investigation and their conclusions. The instructor should present the formal definitions for concave up, concave down, inflection point, and will connect these terms to the sign of the second derivative.

Suggested Follow Up:

- The instructor can assign several exercises for group practice in the classroom.
- The instructor should assign a set of homework problems, from which one can be selected to appear on a short quiz at the beginning of the following class period.

Brief Calculus Activity: **Modeling Real World Optimization Problem**

Purpose: The student will model a real problem and apply the techniques of optimization to obtain a solution.

Background Needed: The student has an understanding of the first and second derivative tests, concavity, critical points, inflection points, and global extrema.

Course Goals Addressed:

- will communicate clearly in written English and be able to comprehend, analyze, and interrogate a variety of written texts
- will communicate orally to be an effective member of the group
- will perform basic mathematical manipulations,display facility with the use of mathematics in framing concepts for mathematical analysis, and interpret data intelligently
- will understand some of the phenomena comprehended by the physical/or life sciences and participate in the experimental process for verifying scientific hypotheses
- will be able to use computers to perform subject-specific tasks appropriate to their major fields

Criteria for Evaluation:

Each student will demonstrate:

1 participation with the group
2 understanding of the statement of the problem
3 solutions written in clear complete sentences
4 accurate use of mathematical notation
5 solutions supported with previously presented concepts and accurate computation
6 coherent oral presentation with proper audio visual aids (if needed)

Instructions: As a group, formulate a solution to the following design problem. Make sure your solutions are written in clear complete sentences, your answers are supported with concepts previously presented, and that your computations are correct.

You work for a container company, and your boss finds out you had Brief Calculus. He brings you a 3 cubic foot shipping carton, and he wants to know if there is a way to make a carton with the same volume which uses less cardboard. The carton has to be made from a single rectangular sheet of cardboard. It has to have a square base, and the top and bottom flaps must fold to make the top and bottom of the carton double-layered, just like the one he brought you. You want to get ahead, so you decide to determine the absolute minimum amount of material required to make a box according to the specifications.

From Brief Calculus, you remember that you should:

(1) Draw a picture which clearly relates the dimensions of the box to the dimensions of the material used to make it.

(2) Write a formula for the amount of material used to make the box.

(3) Re-write this formula so that it is a function of only one variable, and determine the largest and smallest values the variable can have.

(4) Use what you know about finding extrema to solve the resulting minimization problem.

(5) Check the box the boss brought in to see if it is the best size.

Prepare a presentation using audio-visual aids to convince your boss(the class) that your solution minimizes the amount of cardboard needed.

To the Instructor:

Hints and Comments:

At the end of the presentations the class will critique each group's presentation. After the presentations the instructor may give a summary of the way to approach and work this problem if the students solutions are not clear.

Brief Calculus Activity: **Compound Interest and the Definition of e**

Purpose of Activity: Discovering the number e through numeric and graphical investigation of continuous compounding.

Background Needed: The student has had exposure to the formula for the balance in an account under compound interest:

$$A(t) = A_0(1 + r/n)^{nt}$$

where A_0 is the amount of the original investment, n is the number of times per year that interest is paid,r is the annual interest rate (expressed in decimal form), and t is the time in years the money has been invested. Experience using technology to graph functions, and to zoom in (using technology) on a point on a graph to find its coordinates accurate to a desired number of decimal places is necessary. Some previous experience working informally with mathematical limits will be helpful.

Course Goals Addressed:

- Develop mathematical modeling ability to solve real-world problems
- Demonstrate the ability to communicate effectively in mathematics to various audiences
- Use concepts and techniques of calculus to analyze functional relationships
- Demonstrate the mathematical manipulative skills necessary for problem solving
- Use graphs to help conceptualize and solve problems.

Criteria for Evaluation:

Each student will demonstrate:

1) Correct use of English
2) Composition in complete sentences
3) Correct and appropriate use of mathematical notation
4) Clear and coherent explanations exhibiting appropriate logical development
5) Accurate work and correct answers
6) Use of standard English in oral presentation
7) Organized presentation within a logical framework
8) Effective use of clear and legible visual aids (such as transparencies or boardwork)
9) Presentation with appropriate pacing
10) Appropriate and correct mathematical content
11) Appropriate connection of mathematics with real-world application

Instructions:

Senator Foghorn receives a large salary increase from Congress (which he of course voted against). He wants to "beef up" his retirement account in the Taxpayers' National Bank. The Bank agrees to pay him 100% interest per year, compounded as often as he would like. (The Senator is a special customer: his brother-in-law is the Bank's President.) The honorable Senator has not taken Brief Calculus, so he employs you to advise him. You suggest that you first concentrate on the effect of how frequently interest is paid. To isolate this effect, you prepare a report for the Senator (based on your knowledge of compound interest) on the balance he would have in his account after one year if he invested just $1.00 at the agreed upon 100% annual interest rate, with interest compounded at various numbers of times per year.

You know that for compounding n times per year, the investment at a 100% rate ($r = 1.00$) has a value after t years given by the formula

$$A(t) = A_0(1 + 1/n)^{nt}$$

You also use the facts that in this situation, $A_0 = 1$ (one dollar) and $t = 1$ (one year), and so the compound interest formula becomes

$$A(1) = (1 + 1/n)^n$$

Thus mathematically armed, you prepare for the Senator the following table, using a calculator or software, to show the value of the $1.00 investment after one year.

Your job now in Brief Calculus is to show your classmates and your instructor how you report to the Senator.

1. Using a calculator or computer, verify the results below and fill in the two blanks.

n		
1	$(1 + 1/1)^1$	$ 2.00 with annual compounding
2	$(1 + 1/2)^2$	$ 2.25 with semi-annual compounding
4	$(1 + 1/4)^4$	$ _ with quarterly compounding
12	$(1 + 1/12)^{12}$	$ 2.61 with monthly compounding
365	$(1 + 1/365)^{365}$	$ _ with daily compounding
8760	$(1 + 1/8760)^{8760}$	$ 2.72 with hourly compounding
525600	$(1 + 1/525600)^{525600}$	$ 2.72 with compounding each minute
31536000	$(1 + 1/31536000)^{31536000}$	$ 2.72 with compounding each second

You had to explain to the Senator that the number 8760 used for hourly compounding was

(24 hours/day) x (365 days/year),

the number of hours per year, and that in a similar way you computed that there are 525600 minutes per year and 31536000 seconds per year.

2. Verify that there are 31536000 seconds per year.

The Senator is puzzled that the balance in his account levels out at $2.72, despite increases in the number of times per year that interest is paid. He wants to know why he stops getting better interest even when the compounding period is shortened dramatically. After all, he correctly observes, the more often interest is paid, the more it happens that interest already paid is itself earning new additional interest; "So," the Senator demands, "why don't I continue to earn dramatically more interest when the compounding period continues to be dramatically shortened?" To help you explain this curious phenomenon to the Senator, you complete the steps outlined in the remainder of this activity, drawing on your knowledge of functions, graphing, and finding limits, which you learned in Brief Calculus.

You know from previous work that for compounding n times per year, the investment at a 100% rate has a value after t years given by the formula

$$A(t)=A_0(1+1/n)^{nt}=A_0[(1+1/n)^n]^t$$

Hence for compounding continuously (roughly speaking, compounding once each moment), the investment has a value after t years given by the formula

$$A(t)=A_0[\lim_{n\to\infty}(1+1/n)^n]^t$$

Thus it turns out that you need to find the value of the expression $(1 + 1/n)^n$ as n becomes larger and larger and larger. Let us investigate this using a calculator or computer.

3. Verify the entries given in the third column of the following table, and then complete the table with values accurate to five decimal places (instead of just the two places you used in your original report), using a calculator or software, to show the value of the $1.00 investment after one year. Of course you should use that in this situation, $A_0 = 1$ (one dollar) and $t = 1$ (one year), and so the compound interest formula becomes
$A(1) = (1 + 1/n)^n$.

Compounding Period	n	$A(1) = (1 + 1/n)^n$
Year	1	2.00000
Half-year	2	
Quarter	4	
Month	12	2.61304
Day	365	
Hour	8760	
Minute	525600	
Second	31536000	

The way the numbers in the third column in the above table settle down toward a fixed value even as the compounding period gets shorter and shorter enables us to give meaning to the notion of continuous compounding, the key to our explanation to the Senator.

There is however a problem with the approach of taking larger and larger values of n to estimate the value we need in making our report to the Senator: it is difficult to know how large to take the number n to get an estimate of the limiting value we seek with (let us say) five decimal place accuracy, and the Senator wants to know how much he can squeeze out of the system down to the penny. So for job security, we employ a standard mathematical technique: We replace an investigation involving $n \to \infty$ (the number of annual compounding periods grows large without bound) with an investigation involving $1/n \to 0$ (the length of a compounding period shrinks to zero).

First notice that if you replace n by $x = 1/n$ in the expression

$$(1 + 1/n)^n,$$

you obtain the expression

$$(1 + x)^{(1/x)},$$

since $x = 1/n$ implies that $n = 1/x$. Moreover, as $n \to \infty$, $x = 1/n \to 0$.

So

$$\lim_{n\to\infty}(1+1/n)^n = \lim_{x\to 0}(1+x)^{1/x}$$

4. This second limit above, $\lim_{x \to 0} (1 + x)^{(1/x)}$, cannot be evaluated simply by substituting $x = 0$ into the expression $(1 + x)^{(1/x)}$. Explain why not.

So instead we shall use zoom-in with a graphics calculator or software as described below.

5. Use a graphics calculator or software to draw the graph of

$$y=(1+x)^{1/x}$$

in an appropriate viewing rectangle which includes $x = 0$. (Notice that the function is undefined at $x = 0$. However your graph will correctly suggest that the function nonetheless has a limiting value as $x \to 0$.) Sketch the graph here.

6. Now zoom-in on the part of the graph near $x = 0$ to determine the limit as $x \to 0$; be sure to zoom in far enough so that you can be sure that you have obtained the limiting value accurate to five decimal places. (If you don't know how to be sure of this, ask your instructor).

$\lim_{x \to 0} (1 + x)^{(1/x)} =$ (numerical value of the limit)

7. Do you recognize this limit from the decimal value you just obtained? If so, what is this number usually called? (If you don't know the standard mathematical symbol for this limit, find out from some resource such as a classmate, your textbook, or your instructor.)

$\lim_{x\to 0} (1 + x)^{(1/x)} =$ (symbol for the value of the limit)

8. Next look back to an early part of this activity to remind yourself of the connection between the limit in the question just above and the limit just below, and use this connection to complete the equation below.

$\lim_{n\to\infty}(1 + 1/n)^n =$ (symbol for the value of the limit)

9. Rewrite the formula for compound interest with continuous compounding from the discussion just above question 3, replacing the value of the limit involved with the symbol from the part just above this which represents the value of this limit.

$A(t) =$

10. If Senator Foghorn invests $1000, use the five-decimal place accuracy for the number you obtained in this activity to find his annual return in dollars and cents if interest is compounded continuously.

$A(1) =$

11. Homework This is to be done individually or (preferably) in groups, as specified by your instructor. Your instructor will announce the due date.

On a separate page, write a brief explanation to the Senator explaining what happens as the number of compounding periods keeps increasing. Include in your explanation why the return on his investment of $1.00 levels off at $2.72, and explain what would happen if instead of investing $1.00 he invested a larger amount such as $10.00, $100, or $1000.00.

Historical Note: The number you have just obtained as a limit arises in many real-world settings, not just in continuous compound interest. This number is called *e*, in honor of the Swiss mathematician Leonard Euler (pronounced "Oiler") who lived from 1701 to 1783. Euler was the first to recognize and establish the importance of this limit.

If you want to study the decay of radioactive materials in the atmosphere or the growth of bacteria in a culture, you will need the number *e*. *e* is an irrational constant real number which occurs in nature. In that way, *e* is similar to the number π. One way to think of *e* is to remember that $1 compounded continuously for one year at a rate of 100% will turn into *e* dollars. Notice that *e* is not a variable. It is a constant, but since it is irrational, *e* cannot be expressed as a terminating or a repeating decimal. *e* lives on the number line between 2 and 3, closer to 3 than to 2. You will locate *e* on a number line in the next exercise.

The number *e* is the base of the natural exponential function, $y=e^x$, which plays a central role in applications of mathematics. If we interchange the x and y coordinates in all the ordered pairs which belong to the natural exponential function, we get the natural logarithm function $y=\ln x$ (the notation "ln" stands for "natural logarithm"). Calculus operations involving logarithmic and exponential functions are greatly simplified when *e* is used as a base. You can use any scientific calculator to get approximate values for e^x and $\ln x$. These functions will have enormous significance in your study of calculus and will arise in a wide variety of real-world applications.

12. Since the number e is so important in mathematics and its applications, it is important for us to fix in our minds the approximate value of e. To help fix the numerical value of e in your mind, mark the approximate location of e on the number line below, and label your point with the symbol e below the line.

```
----|----|----|----|----|----|----|----|----|--->x
   -4   -3   -2   -1    0    1    2    3    4
```

13. Since the function $y = e^x$ is so important, you should learn its graph. Use your graphics calculator or software to draw the graph of $y = e^x$. Sketch this graph below to help fix the graph in your visual memory.

14. Use the above graph of $y = e^x$ to complete the following:

(a) The domain of the function $y = e^x$ is ____________________.

(b) The range of the function $y = e^x$ is ____________________.

Topics for further exploration

1. Can you explain algebraically (without reference to the graph of $y = e^x$) why the exponential function based upon e cannot produce zero or a negative number as an output value?

2. In the above work, we used an annual interest rate of 100%. This made the mathematical work simpler, but (the story about Senator Foghorn notwithstanding) this is really not an interest rate on an investment that most of us can ever hope to encounter. Go through this work again, starting with the limit transformation discussion preceding part 4, but this time using interest rate r (without assigning a numeric value to r). HINT:

$$\lim_{n\to\infty}(1+r/n)^n = \lim_{n\to\infty}[(1+r/n)^{n/r}]^r$$

Use your result to find the balance after five years if \$20,000 is invested at 8% annual interest.

To the Instructor:

Hints and Comments:

We suggest that the students should receive this handout in advance of the class devoted to it, and that the students should read page 1 and work down through problem three before class, either individually or (preferably) in groups. Students will work in small groups in class to produce written answers to the problems beyond #3. One written report per group will be turned in at the end of class. Problem #11 is to be done outside of class, either individually or (preferably) in groups, as assigned by the instructor. One or more of the small groups may be asked to present their solutions to the class, with class discussion following.

Brief Calculus Activity: **Logistic Curves**

Purpose: The student will discover and study a mathematical model for growth under limiting constraints.

Background Needed: The student has familiarity with exponential growth (growth without limiting constraints) and some experience relating graphs and verbal descriptions of varying quantities. An understanding of the concepts of increasing and decreasing functions, and of concavity and how concavity of a graph relates to change in the rate of change and an understanding of inflection points is necessary. Familiarity with graphing technology is also important.

Course Goals Addressed:

- Develop mathematical modeling ability to solve real-world problems
- Demonstrate ability to communicate effectively in mathematics to various audiences
- Use concepts and techniques of calculus to analyze functional relationships
- Use graphs to help conceptualize and solve problems
- Think qualitatively using calculus concepts about quantitative information

Criteria for Evaluation:

Each student will demonstrate:

1) Correct use of English
2) Composition in complete sentences
3) Correct and appropriate use of mathematical notation
4) Clear and coherent explanations exhibiting appropriate logical development
5) Accurate work and correct answers

Instructions:

1. Use a graphics calculator or software to find a complete graph of

$$y=\frac{200}{1+9e^{-0.8t}}$$

and

$$y=20e^{0.8t}$$

in the same viewing rectangle (window). Sketch these graphs below.

a. Which one of the above graphs describes unconstrained growth?

b. Which one of the above graphs describes constrained growth?

c. Use your graphing utility to find the limit as $t \to \infty$ of the function which describes constrained growth. (For example, with a graphing calculator you can trace.) Record your result here.

Two situations involving growth over a period of time are described below. You are to answer questions related to each situation. In both cases you are to determine the basic shape of a graph which models the situation.

<u>First situation</u>: A biology student introduces a small number of cockroaches into a laboratory container. The student supplies plenty of food and water, but space restrictions will come into play eventually. Experience shows that the population will increase over time. Initially the population is so small relative to the size of the container that the containment imposes no limitations on growth. However, the size of the container will eventually begin to restrict and later severely restrict the growth of the population of cockroaches; in fact, there is a maximum number of cockroaches the container can support.

Let $P(t)$ be the population of cockroaches t days after the original population is introduced into the container. In the next few questions, you will reason out the basic shape of the graph of $P(t)$.

2. The population $P(t)$ of cockroaches increases over time. What does this imply about the graph of $P(t)$?

3. Sketch the graph of P(t) for a period of time before the size of the container imposes any restrictions. What does your previous experience with unrestricted growth suggest about the shape of the graph during this time period?

4. When space restrictions begin to come into play, how does the shape of the curve change from the shape in the portion of the graph sketched in problem 3? What calculus term describes the point where the shape changes?

5. What can you conclude about the graph of P(t) from the fact that there is a maximum number of cockroaches the container can support? Call this maximum number M; mark an appropriate point on the y-axis in problem 3 and label it M.

(Note: M is called the carrying capacity of the environment for this species.)

6. Return now to the graph you started in problem 3 and finish the sketch, taking account of the observations you made in problems 4 and 5.

7. Do you suppose that functions defined by a formula (as in question 1) would model this type of situation? By trial and error, choose numbers M, A, and B such that the graph of

$$y = \frac{M}{1 + Ae^{Bt}}$$

models the behavior of this cockroach population. Use a graphics calculator or software to graphically check the equation you write. If the graph is not what it should be, modify your equation and try again. Write the equation you finally decide upon. (Hint: In the equation for constrained growth in problem 1, how do the value of y when $t = 0$ and the limiting value of y show up in the equation?)

Second situation: Retail sales of some "fad" items sometimes follow the same growth pattern we found above for the cockroach population.

8. Describe in words how the weekly sales would vary as the weeks go by, and also sketch a possible graph for accumulated total sales as a function of time in weeks. Hint: Let S(t) be the accumulated total number of items sold t weeks after the item is introduced; thus S(1) is the number of items sold in the first week and S(2) is the number of items sold in the first week plus the number sold in the second week. The graph of the sales function S(t) is similar to the graph of P(t).

9. Two major firms both sold a particular brand of jeans. At some point in time, both firms placed an order to the manufacturer. One of the firms had noticed by way of a spreadsheet analysis of the past week's sales that, although sales of jeans continued to be brisk, the rate of sales per week was decreasing, and so placed a smaller order. The other firm placed its usual large order.

Which point on the graph of $S(t)$ corresponds to the point in time when sales first begin to increase at a decreasing rate? What is the calculus name for this type of point?

10. Describe in words the likely consequences for each firm of its ordering decision. (Consider issues of overstocking and/or understocking.)

To the Instructor:

Hints and Comments:

The students will need some help in problem #7 when they are trying to estimate the values of the parameters in the equation.

The instructor may try to bring real data to the class for certain species and ask the students to make a graph of the data to verify some of the trends observed in population growth models.

Brief Calculus Activity: **Visualizing Depreciation as a Signed Area**

Purpose: The student will discover how to use definite integrals to visualize the amount of depreciation over a time period. The idea is to use the Fundamental Theorem of Calculus to express a change in value of some function as a definite integral of the derivative of that function, and then to draw a region whose signed area is the value of that integral. We then have a region whose signed area is the change in value of the function. Thus we provide a way literally to see a change in value of a function.

Background Needed: The student has previously encountered straight-line depreciation, has the ability to interpret a definite integral as a signed area, has an understanding of the derivative as a rate of change, has familiarity with determining area of a trapezoid, and has familiarity with computing a definite integral using the Fundamental Theorem of Calculus:

$$\int_a^b F'(t)dt = F(b) - F(a)$$

Course Goals Addressed:

- Develop mathematical modeling ability to solve real-world problems
- Demonstrate the ability to communicate effectively in mathematics to various audiences
- Develop the ability to use calculus concepts to interpret quantitative information qualitively
- Use graphs to help conceptualize and solve problems

Criteria for Evaluation:

Each student will demonstrate:

1) correct use of English
2) writing in complete sentences
3) correct and appropriate use of mathematical notation
4) mathematical understanding through clear and coherent explanations which exhibit appropriate logical development
5) ability to work accurately and to obtain correct answers

Instructions: Accountants use the notion of book value to approximate market value. Book value of a business asset is the difference between purchase price and accumulated depreciation; thus depreciation provides a way to measure book value. Accountants use a variety of depreciation methods; different choices may be made, for example, to achieve tax advantages or to help understand the performance of a business.

In this activity we will investigate two depreciation methods. The first method we study is mathematically simple, and provides an introduction to some techniques of analysis which we can then apply to the second method (double declining balance), which is actually used in practice.

We shall use the notation $V(t)$ to denote the (book) value in dollars of an automobile t years after it was purchased new. Of course, the value $V(t)$ decreases as time t marches on. In problems 1, 2, and 3, you are to find general statements true for any method of depreciation, without assuming any particular formula for $V(t)$.

1. (a) Express the value of the integral below in terms of values of the function V, using the Fundamental Theorem of Calculus. Since you do not have a formula yet for V or V', your answer will have to be expressed in terms of the symbol V.

$$\int_0^1 V'(t)dt =$$

(b) You just wrote the above integral as the difference of two values; that is, the integral was written as a change in the value of the car. Write a sentence which states explicitly this interpretation of the integral. (Be sure to mention the time period involved.)

2. For this problem, you should build on the observations you made in problem 1 above. Write the change in value of the car during its third year of use first as a difference, and then represent that difference as a definite integral.

3. If the car is expected to last six years, write the change in value of the car during its last year of use first as a difference and then as a definite integral.

4. Now let us see how these ideas play out for a specific method of depreciation. Suppose the purchase price of the car was \$20000, and that the car depreciates over six years to a value of \$2000 in such a way that initially, the rate of depreciation is \$6000 per year, and at the end of the six year period the rate of depreciation is \$0 per year. Finally, suppose that the rate of depreciation is a linear function of time. We shall denote the (book) value t years after purchase of the car under this depreciation method by $V_1(t)$.

(a) According to the above information, $V_1(0) = 20000$ and $V_1'(0) = -6000$. Similarly,

$V_1(6) =$ __________ and $V_1'(6) =$ __________.

(b) Sketch the graph of the function $y = V_1'(t)$ below by first plotting the points $(0, V_1'(0))$ and $(6, V_1'(6))$, and then using the given information that $y = V_1'(t)$ is a linear function. (You will need to recall the type of graph that a linear function has.)

(c) Before continuing, be sure you understand why this method of depreciation is <u>not</u> straight line depreciation. What graph is a straight line if we are using straight line depreciation?

(d) In the sketch you drew in part (b), shade the region whose signed area is the value of the definite integral

$$\int_0^1 V_1'(t)dt.$$

Since this integral evaluates to $V_1(1) - V_1(0)$, the change in value of the car in its first year is the signed area of the region you just shaded.

Now for each of the integrals from problems 2 and 3, shade in the graph of part (b) a region whose signed area is the value of the integral.

(e) Briefly explain by referring to the sizes of the three shaded regions you just drew whether the amount of depreciation (using the depreciation method described in this problem) is different or the same in the first, third, and last years. If the depreciation differs from year to year, include in your description of how the depreciation varies from year to year whether the car depreciates more rapidly at the beginning or at the end of the life of the car.

5. You found in problem 4 that the car depreciates most rapidly at the beginning of its life under the linear rate of depreciation we used in that problem. Exponential depreciation is another method which also involves faster decline in value at the beginning. One form of exponential depreciation which accountants sometimes use is the double-declining balance method. If this method is used over N years, the original value $V_2(0)$ is depreciated each year by $100(2/N)$ percent of its value at the beginning of that year. It turns out that this means that the value $V_2(t)$ after t years is

$$V_2(t) = V_2(0)[(1-2/N)^t].$$

Recall that our car has a purchase price of \$20000, and that the useful life of the car is six years. So:

$V_2(0) =$ ________________ and $N =$ ____________________.

Now write the double declining balance formula for our automobile, using the specific values we are given for $V_2(0)$ and N.

$V_2(t) =$

6. Using graphing technology, find the graph of $V_2'(t)$ for the function $V_2(t)$ you found in problem 5. (A graphing calculator will graph V_2' even if you don't know how to find a formula for V_2'. If you don't know how to make this graph, ask your instructor.) Sketch the graph of $y = V_2'(t)$ for $0 \leq t \leq 6$ below.

7. As you did with the first depreciation method above, write integrals for the change in value in the first year, the third year, and the last year. Then sketch and shade a region in the graph in problem 6 for each of these three integrals whose signed area is the value of the integral, and thus is the change in value of the car. As you did with the first depreciation method, briefly explain by referring to the shaded regions you just sketched how the amount of depreciation varies from year to year using the double-declining balance method.

8. You've now investigated two methods of accelerated depreciation. That is, with both methods of depreciating that we considered above, the value of the car declines most rapidly in the first years of its life. In this part you will make a comparison between the two methods to see which method accelerates the depreciation the most. We will need accurate graphs to make this comparison, so we will use graphing technology to achieve the accuracy we need.

(a) Use a graphics calculator or computer software to graph in the same viewing rectangle both $y = V_1'(t)$ and $y = V_2'(t)$. That is, graph together in the same viewing rectangle the graphs you made in part 4(b) and 6.

The straight line could be graphed by first finding a formula for the function to be graphed and then using a function grapher. However, since we do not otherwise need this formula, you may instead use a command that draws a line segment between the two given points, just as we did when we sketched the line by hand.

(b) Next, use an appropriate command on your graphing utility to draw appropriate portions of the vertical lines $x = 1$, $x = 2$, $x = 3$, $x = 5$, and $x = 6$.

(c) Sketch the results of (a) and (b) below.

(d) Which of the two depreciation methods leads to more depreciation in the first year? Explain briefly.

(e) Which of the two depreciation methods leads to more depreciation in the last year? Explain briefly.

9. (a) Suppose you use your car for business and get a tax write-off. Which of these two methods of depreciation would you prefer to be able to use in the first year? Explain why.

(b) Suppose you want to sell your car at the end of the second year. Which of these two methods of depreciation would you prefer to be able to use to determine book value for your car? Explain why.

For further exploration

In the above activities we deliberately did not find a formula for $V_1'(t)$ or for $V_2'(t)$, and we deliberately did not find the numerical values of the various depreciations involved. This was to emphasize that we can make qualitative comparisons by a visual inspection of the sizes of regions, without having a numerical measure of these sizes. However, you may wish to also compute these. An outline for doing this follows.

10. Find a formula for $V_1'(t)$ (the depreciation method described in problem 4). You could do this by following the procedure outlined below. Refer back to the graph you drew in 4(b) as you find the slope and y-intercept of the line.

(a) Find the slope of the line which is the graph of $y = V_1'(t)$:

(b) Find the y-intercept of the line which is the graph of $y = V_1'(t)$:

(c) Find a formula for $V_1'(t)$. Be sure your formula produces values with the correct sign. That is, you know the value of the car is decreasing; make sure the sign of the rate of change of value is consistent with decreasing value.

11. For the method of depreciation described in problem 4, find numerically the changes in value in the first, third, and last years. Do this in two ways: first by computing directly (using only elementary geometry) the areas which represent these changes in value, and then by symbolically integrating over appropriate intervals the function $V_1'(t)$ you found in problem 10. Recall that the area of a trapezoid is the base times the average height.

12. Find the numerical values of the changes in value for the method of depreciation described in problem 5 in the first, third, and last years. Do this by using your calculator to evaluate the function $V_2(t)$ between appropriate limits.

To the Instructor:

Hints and Comments:

In problem 12, does it make sense to ask the student to use a calculator to numerically integrate the numerical derivative to get a change in value?

Brief Calculus Activity: **Applications of the Definite Integral to Distribution Functions**

Purpose: The student will experience an important application of integration which will give insight and understanding for later work in probability and statistics.

Background Needed: The student has been exposed to the integral as the limit of approximating rectangles. The student can evaluate definite integrals in closed form using the Fundamental Theorem of Calculus and numerically using calculator or computer technology.

Course Goals Addressed:

- Develop mathematical modeling ability to solve real world problems
- Demonstrate ability to communicate effectively in mathematics to various audiences
- Develop the ability to use calculus concepts to interpret quantitative information qualitatively
- Use graphs to help conceptualize and solve problems
- Demonstrate the mathematical manipulative skills necessary for problem solving

Criteria for Evaluation:
Each student will demonstrate

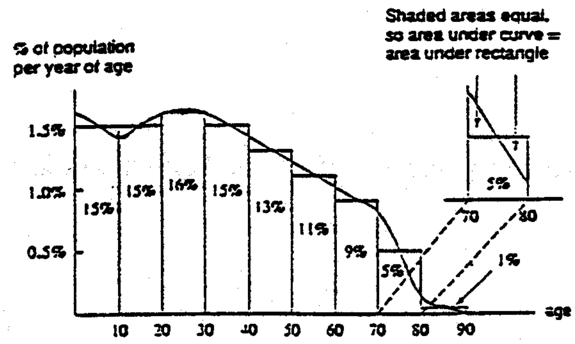
1) understanding of the statement of each problem
2) the logical steps needed for solution
3) appropriate use of standard English and mathematical notation
4) facility with graphing and numerical integration applications with calculator or computer

(This activity is adapted from the Harvard calculus project.)

1. The following is a table showing ages in the US and a graph showing the same data.

Ages in the US in 1990

Age	% of total
0-10	15%
10-20	15%
20-30	16%
30-40	15%
40-50	13%
50-60	11%
60-70	9%
70-80	5%
80-90	1%



Smoothing out the age histogram

Notice that the curve helps us estimate more detailed information than the histogram. The curve is an example of a distribution function.

a) Under the curve, shade the area which represent the percentage of the population between 17.3 and 20.2 years of age.

b) Give a rough numerical approximation of your answer in part (a).

2. Suppose you work for Southern Bell collecting data on phone calls. You determine the distribution function p(x) for the duration of telephone calls within Columbia is

$$p(x)=0.4e^{-0.4x}$$

where x denotes the duration in minutes of a randomly selected call.

a) Graph this function on your calculator.

b) What percentage of calls last between 1 and 2 minutes?

c) What percentage of calls last 1 minute or less?

d) What percentage of calls last 3 minutes or less?

e) Using the answer in part (c), what percentage of calls last 3 minutes or more?

3. A special type of distribution function is the normal distribution function. The mean gives an average and the standard deviation gives an idea of "spread."

One example of a quantity which is normally distributed is the annual rainfall in Anchorage, Alaska. From records kept over the years, meteorologists have determined that the mean of the rainfall distribution is 15 inches and the standard deviation is 1 inch. You will see its distribution function when you draw

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-15)^2/2}$$

where x represents the number of inches of rain per year.

Hint: On your TI calculator this function will look like $y1 = (1/\sqrt{(2\pi)})e\wedge(-(x-15)^2/2)$

A recommended viewing window would be $10 \leq x \leq 20$ and $-1 \leq y \leq 2$.

a) What percentage of the time (i.e. what fraction of years) is rainfall between 14 and 16 inches?

b) What percentage of the time is rainfall between 13 and 17 inches?

Note: Use your calculator to get approximate answers. The Fundamental Theorem of Calculus cannot be used with this function. Why not?

Notice on the graph two points of inflection.

c) Use your calculator to find the x-value of each of these.

d) Find the area under the curve between the two inflection points.

e) Find the percentage of the area under the curve which lies within one standard deviation of the mean.

f) Explain why your answers to (d) and (e) are the same.

4. It is shown in statistics that a normal curve with a mean of 0 and a standard deviation of s is given by:

$$p(x) = \frac{e^{\frac{-x^2}{2s}}}{s\sqrt{2\pi}}$$

We will explore the effect of changing the standard deviation. Use your calculator and the formula above to produce graphs with the following standard deviations, s:

a) standard deviation, s = 1

b) standard deviation, s = 2

c) standard deviation, s = 3

d) For the graph in part (c), use your calculator to find inflection points.

e) Find the area under the curve between the inflection points.

f) Can you predict a "rule of thumb?" If a quantity is normally distributed, about ______% will fall within one standard deviation of the mean.

5. Circle the quantities below which you believe would be normally distributed:

a) IQ's

b) SAT scores

c) NBA shoe sizes

d) altitude of major US cities

e) annual income of US households

f) number of years per marriage in the US

g) the time it takes to finish an exam

Notes to the Instructor:

Hints and Comments

The distribution functions in this activity are from real-world settings and should be appealing to the student. They represent an application of integration not usually seen in Brief Calculus which will be relevant and natural to the students.

This activity is based on an intuitive approach to probability density functions, generically referring to them as distribution functions. You may wish to use "probability density function" and give a formal definition. The intention here is to convey through informal discussion the important principle that the total area under the curve equals 1. With either approach, this activity will give students a basic understanding of bell-shaped curves, and preparation for later work in probability and statistics. In fact, you may wish to follow this activity with further study of the mean, standard deviation, and other statistical concepts. The activity asks the students to find the percentage within one standard deviation of the mean. A natural extension would be to ask for the percentage within two standard deviations of the mean. Here you may also wish to use the term "probability" rather than percentage.

A few suggestions for successful implementation of this activity:

a. A preliminary class discussion on smoothing out the histogram would be a good warm-up before the students begin to work in groups.

b. The students will encounter "impossible" integrals (i.e. those which cannot be evaluated in closed form, using the Fundamental Theorem of Calculus) in this activity. If they have no prior experience evaluating such integrals numerically on their calculators, they will need instruction on this technique.

c. Problem 2e seems to involve an improper integral. It is possible to avoid this concept if students understand that the total area is 1. The answer to 2e can be obtained by subtracting the answer for 2d from 1.

A suggested time table: Plan to devote one class period to this activity. Distribute a copy to each student in the previous class and assign as reading. Begin the activity with a discussion of histograms in which areas represent percentages (or probabilities) and explain the rationale of "smoothing out the histogram." Have the students work in small groups on the given exercises, with periodic large group discussion. Assign the unfinished problems as homework. Students could submit the finished activity individually. Feedback to the students could be in the form of a homework grade or as a take-home quiz score, possibly accompanied by written comments.

Resource List

Calculus Reform Materials

Calculus for a New Century, MAA Notes No. **8**, Mathematical Association of America, 1529 Eighth St., NW, Washington, DC 20036, 1988. A report on the 1987 Symposium at Tulane University sponsored by the National Science Foundation, which focused national attention on the need for reform in the teaching of calculus.

J. Callahan, D. Cox, *et. al.*, *Calculus in Context*, The Five College Project, W. H. Freeman, New York, 1993.

Thomas P. Dick and Charles M. Patton, *Calculus*, The Oregon State Project, PWS-Kent, 20 Park Plaza, Boston, MA 02116, 1993.

Deborah Hughes-Hallet, Andrew M. Gleason, *et. al.*, *Calculus*, The Harvard Project, John Wiley & Sons, New York, 1994.

A. Osterbee, and P. Zorn, *Calculus from Graphical, Numerical and Symbolic Points of View*, The Saint Olaf College Project, Saunders College Publishing, 1992.

David A. Smith and Lawrence C. Moore, *The Calculus Reader*, with lab manual, Duke's Project CALC, D. C. Heath, College Division, 125 Spring St., Lexington, MA 02173, 1992.

References for Laboratory Projects

Marcus Cohen, Edward Gaughan, *et. al.*, *Student Research Projects in Calculus*, The Mathematical Association of America, 1529 Eighth St., NW, Washington, DC 20036, 1991.

Resources for Calculus, volume 4: Problems for Student Investigation (Michael Jackson and John Ramsay, eds.), MAA Notes **30**, The Mathematical Association of America, 1529 Eighth St., NW, Washington, DC 20036, 1993.

Resources for Calculus, volume 1: Learning by Discovery (Anita Solow, ed.), MAA Notes **27**, The Mathematical Association of America, 1529 Eighth St., NW, Washington, DC 20036, 1993.

S. Hilbert, *et. al.*, *Calculus, An Active Approach with Projects*, The Ithaca College Calculus Group, Wiley, 1994.

Self Assessment

Group Number:__________

Name:_____________________

1. To what extent did you participate with and contribute to the group?

Low High
|_________|_________|_________|_________|_________|

Comments:

2. To what extent did your partners participate and contribute to the group?

Low High
(1) |_________|_________|_________|_________|_________|

(2) |_________|_________|_________|_________|_________|

(3) |_________|_________|_________|_________|_________|

Comments:

3. Was the material in this activity understandable based on your prior knowledge?

No Yes
|_________|_________|
Comments:

4. What do you think was the most important concept in this activity?

5. What concept, if any, is still confusing to you after this activity?

Student Presentations - Evaluation Form

Name____

Date________________________________

On a scale from 1 to 5 (five being the highest), rate the presentation on

Accuracy of solution___________________

Legibility of written work______________

Projection/clarity of voice_____________
(Could you hear the speaker clearly?)

Organization/clarity of explanation_______
(Did you understand the solution?)

Defense of solution_____________________
(Did the speaker convince you he/she was correct?)

Overall quality of presentation__________

Do you have any suggestions or comments for the speaker?